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A Note on g-Closure and g-Interior

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Abstract

In this paper we have obtained significant properties of closure and interior of a set in generalized topological spaces.

Keywords: Generalized Topological Space, Generalized Closure, Generalized Interior.

Introduction

The notion of generalized topology was introduced by Csaszar [2] in 2002, and he has studied the concept of g-closure and g-interior in generalized topological spaces. In this paper we have obtained significant properties of g-closure and g-interior. We have also constructed some useful examples of g-closure and g-interior in generalized topological spaces.

2. Preliminaries

In this section we recall basic properties of closure and interior of a set in topological spaces.

Definition 2.1

Let X be a non empty set and let τ be a family of subsets of X. Then τ is said to be **topology** on X if following three properties are satisfied viz.;

1. $\emptyset$ and X are in τ,
2. If G_1 and G_2 are elements of τ then $G_1 \cap G_2 \in \tau$,
3. If $G_i \in \tau, for\ i \in I$ then $\bigcup_{i \in I} G_i \in \tau$.

The pair (X, τ) is called **topological space** and elements of family τ are called **open sets** in topological space X. complement of open sets are called **closed sets** in X.

Example 2.1

Let $X = \{x_1, x_2, x_3\}$ and $\tau = \{\emptyset, X, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$. Then τ is a topology on X.

Proposition 2.1

Let (X, τ) be a topological space. Then the following conditions are satisfied:

1. $\emptyset$, X are closed sets in X.
2. Arbitrary intersection of closed sets is a closed set in X.
3. Finite union of closed sets is a closed set in X.

Definition 2.2

Let (X, τ) be a topological space and $A \subseteq X$. Then the **closure** of A is defined as the intersection of all closed sets in X containing A. The closure of A is denoted by $Cl\,(A)$.

Remark 2.1

We note that $Cl\,(A)$ is the smallest closed set in (X, τ) containing A.

Proposition 2.2

Let (X, τ) be a topological space and $A \subseteq X$. Then A is closed if and only if $Cl\,(A) = A$.

Proposition 2.3

Let (X, τ) be a topological space and A, B be subsets of X. Then following properties holds:

1. $Cl\,(\emptyset) = \emptyset$.
2. $Cl\,(X) = X$.
3. If $A \subseteq B$ then $Cl(A) \subseteq Cl(B)$.
4. $Cl(A \cup B) = Cl(A) \cup Cl(B)$.
5. $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$.
6. $Cl(Cl\,(A)) = Cl(A)$.

Proposition 2.4

Let (X, τ) be a topological space and $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of subsets of X. Then

1. $\bigcup_{\alpha \in \Lambda} Cl(A_\alpha) \subseteq Cl\,(\bigcup_{\alpha \in \Lambda} A_\alpha)$.
2. $Cl\,(\bigcap_{\alpha \in \Lambda} A_\alpha) \subseteq \bigcap_{\alpha \in \Lambda} Cl\,(A_\alpha)$..

Definition 2.3

Let (X, τ) be a topological space and $A \subseteq X$. Then the **interior** of A is defined as the union of all open sets in X contained in A. The interior of A is denoted by $Int\ (A)$.

Remark 2.2

We note that $Int\ (A)$ is the largest open set in (X, τ) contained in A.

Proposition 2.5

Let (X, τ) be a topological space and $A \subseteq X$. Then A is open if and only if $Int\ (A) = A$.

Proposition 2.6

Let (X, τ) be a topological space and A, B be subsets of X. Then following properties holds:

1. $Int\ (\emptyset) = \emptyset$.
2. $Int\ (X) = X$.
3. If $A \subseteq B$ then $Int\ (A) \subseteq Int\ (B)$.
4. $Int\ (A) \cup\ Int(B) \subseteq Int\ (A \cup B)$.
5. $Int\ (A \cap B) = \ Int\ (A) \cap Int\ (B)$.
6. $Int\ (Int\ (A)) = \ Int\ (A)$.

Proposition 2.7

Let (X, τ) be a topological space and $\{A_\propto\}_{\propto \in \wedge}$ be a family of subsets of X. Then

1. $\cup_{\propto \in \wedge} Int\ (A_\propto) \subseteq Int\ (\cup_{\propto \in \wedge}\ A_\propto)$.
2. $Int\ (\cap_{\propto \in \wedge} A_\propto) \subseteq \cap_{\propto \in \wedge} Int\ (A_\propto)$.

Proposition 2.8

Let (X, τ) be a topological space and $A \subseteq X$. Then

1. $Int\ (X - A) = \ X - Cl\ (A)$.
2. $Cl\ (X - A) = \ X - Int\ (A)$.

3. g-Closure and g-Interior

In this section we have studied notions of g-closure and g-interior in generalized topological spaces and obtained their significant properties. Further we have obtained useful examples related to this context.

Definition 3.1 [2]

Let X be a non-empty set and let τ_g be a family of subsets of X. Then τ_g is said to be generalized topology on X if following two properties are satisfied viz.;

1. $\emptyset, X \in \tau_g$,
2. If $G_\lambda \in \tau_g$ for $\lambda \in \wedge$ then $\cup_{\lambda \in \wedge} G_\lambda \in \tau_g$.

The pair (X, τ_g) is called **generalized topological space**.

The elements of family τ_g are called **g-open sets** and their complements are called **g-closed sets**.

Example 3.1

Let us consider set $X = \{x_1, x_2, x_3\}$. Then we see that $\tau_g = \{\emptyset, X, \{x_1, x_2\}\{x_2, x_3\}\}$ is a generalized topology on X but is not a topology on X. Thus (X, τ_g) is a generalized topological space but is not a topological space.

Proposition 3.1

Let (X, τ_g) be a generalized topological space. Then the following conditions are satisfied:

1. $\phi\ and\ X$ are g-closed sets in X.
2. Arbitrary intersection of g-closed sets is g-closed set in X.

Proof

1. Since $\phi\ and\ X$ are g-open sets, it follows that their complement $X\ and\ \phi$ are g-closed sets in X.
2. Let $\{F_\propto\}_{\propto \in \wedge}$, where $\wedge$ is an index set, be a family of g-closed sets in X. Now $X - \cap_{\propto \in \wedge} F_\propto = \cup_{\propto \in \wedge}\ (X - F_\propto)$. Since each $X - F_\propto$ is a g-open set in X and being arbitrary union of g-open sets $\cup_{\propto \in \wedge}\ (X - F_\propto)$ is a g-open set in X. Hence $X - \cap_{\propto \in \wedge} F_\propto$ is a g-open set in X. Thus $\cap_{\propto \in \wedge} F_\propto$ is a g-closed set in X.

Remark 3.1

We note that union of two g-closed sets in X may not be a g-closed set in X.

Definition 3.2 [2]

Let (X, τ_g) be a generalized topological space and $A \subseteq X$.Then the **g-closure** of A is defined as the intersection of all g-closed sets in X containing A. The g-closure of A is denoted by $c_g(A)$.

Remark 3.2

We note that $c_g(A)$ is the smallest g-closed set in (X, τ_g) containing A.

Proposition 3.2

Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Then A is g-closed set if and only if $c_g(A) = A$.

Proof

Let A be a g-closed set in X. Then clearly the smallest g-closed set containing A, is itself A. Hence $c_g(A) = A$. Conversely suppose $A \subseteq X$ and $c_g(A) = A$. Since $c_g(A)$ is a g-closed set in X, it follows that A is g-closed set in X.

Proposition 3.3

Let (X, τ_g) be a generalized topological space and let A, B be subsets of X. Then following properties holds:

1. $c_g(\phi) = \phi$, $c_g(X) = X$.
2. If $A \subseteq B$ then $c_g(A) \subseteq c_g(B)$.
3. $c_g(A) \cup c_g(B) \subseteq c_g(A \cup B)$.
4. $c_g(A \cap B) \subseteq\ c_g(A) \cap c_g(B)$.
5. $c_g\left(c_g(A)\right) = c_g(A)$.

Proof

1. Since ϕ and X are g-closed sets, from Proposition 3.2, we have, $c_g(\phi) = \phi$ and $c_g(X) = X$.
2. Suppose $A \subseteq B$ in X. Since $B \subseteq c_g(B)$ and $A \subseteq B$, we have $A \subseteq c_g(B)$. Now $c_g(B)$ is a g-closed set and $c_g(A)$ is the smallest g-closed set containing A, we find that $c_g(A) \subseteq c_g(B)$.
3. Since $A \subseteq A \cup B, B \subseteq A \cup B$ from (ii) we have $c_g(A) \subseteq c_g(A \cup B)$ and $c_g(B) \subseteq c_g(A \cup B)$. This implies $c_g(A) \cup c_g(B) \subseteq c_g(A \cup B)$.
4. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ from (ii) we have $c_g(A \cap B) \subseteq\ c_g(A)$ and $c_g(A \cap B) \subseteq c_g(B)$. This implies $c_g(A \cap B) \subseteq\ c_g(A) \cap c_g(B)$.
5. Since $c_g(A)$ is a g-closed set in X, it follows that $c_g\left(c_g(A)\right) = c_g(A)$.

In the above Proposition 3.3 (iii) we note that $c_g(A) \cup c_g(B) \neq c_g(A \cup B)$. We have following Example.

Example 3.2

Let us consider set $X = \{x_1, x_2, x_3, x_4\}$ with respect to generalized topology $\tau_g = \{\phi, X, \{x_2, x_3\}, \{x_1, x_2, x_4\}\}$. Then the family of g-closed sets is given by $\tau_g^c = \{\phi, X, \{x_3\}, \{x_1, x_4\}\}$. Let us consider sets $A = \{x_1\}, B = \{x_3\}$. Then $c_g(A) = \{x_1, x_4\}\ and\ \ c_g(B) = \{x_3\}$. Now $c_g(A) \cup c_g(B) =$

$\{x_1, x_3, x_4\}$ and $c_g(A \cup B) = X$. Therefore $c_g(A) \cup c_g(B) \neq c_g(A \cup B)$.

In the above Proposition 3.3 (iv) we note that $c_g(A \cap B) \neq c_g(A) \cap c_g(B)$. We have following Example.

Example 3.3

Let $X = \{x_1, x_2, x_3, x_4\}$ and $\tau_g = \{\phi, X, \{x_1\}, \{x_4\}, \{x_1, x_4\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}\}$ be generalized topology on X. Then the family of g-closed sets is given by $\tau_g^c = \{\phi, X, \{x_1\}, \{x_4\}, \{x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3\}\}$. Let us consider sets $A = \{x_1, x_2\}$, $B = \{x_3, x_4\}$. Then $c_g(A) = \{x_1, x_2, x_3\}$ and $c_g(B) = \{x_2, x_3, x_4\}$. Now $c_g(A) \cap c_g(B) = \{x_2, x_3\}$ and $c_g(A \cap B) = c_g(\phi) = \phi$. Hence $c_g(A \cap B) \neq c_g(A) \cap c_g(B)$.

Proposition 3.4

Let (X, τ_g) be a generalized topological space and $\{A_\propto\}_{\propto \in \Lambda}$ be a family of subsets of X. Then

1. $\bigcup_{\propto \in \Lambda} c_g\ (A_\propto) \subseteq c_g\ (\bigcup_{\propto \in \Lambda} A_\propto)$.
2. $c_g\ (\bigcap_{\propto \in \Lambda} A_\propto) \subseteq \bigcap_{\propto \in \Lambda} c_g\ (A_\propto)$.

Proof

Similar to proof of Proposition 3.3 (iii) and (iv).

Definition 3.3 [2]

Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Then the **g-interior** of A is defined as the union of all g-open sets in *X* contained in *A*. The g-interior of *A* is denoted by $i_g(A)$.

Remark 3.3

We note that $i_g(A)$ is the largest g-open set in (X, τ_g) contained in *A.*

Proposition 3.5

Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Then *A* is g- open if and only if $i_g(A) = A$.

Proof

Let A be a g-open set in X. Then clearly the largest g-open set contained in A, is itself A. Hence $i_g(A) = A$. Conversely suppose $A \subseteq X$ and $i_g(A) = A$. Since $i_g(A)$ is a g-open set in X, it follows that A is a g-open set in X.

Proposition 3.6

Let (X, τ_g) be a generalized topological space and let *A*, *B* be subsets of *X*. Then the following properties hold:

1. $i_g(\phi) = \phi, i_g(X) = X$.
2. If $A \subseteq B$ then $i_g(A) \subseteq i_g(B)$.
3. $i_g(A) \cup i_g(B) \subseteq i_g(A \cup B)$.
4. $i_g(A \cap B) \subseteq i_g(A) \cap i_g(B)$.
5. $i_g\ \left(i_g(A)\right) = i_g(A)$.

Proof

1. Since ϕ and *X* are g-open sets, from Proposition 3.5, we have, $i_g(\phi) = \phi$ and $i_g(X) = X$.
2. Suppose $A \subseteq B$ in X. Since $i_g(A) \subseteq A$ and $A \subseteq B$, we have $i_g(A) \subseteq B$. Now $i_g(A)$ is a g-open set and $i_g(B)$ is the largest g-open set contained in B, we find that $i_g(A) \subseteq i_g(B)$.
3. Since $A \subseteq A \cup B$, $B \subseteq A \cup B$ from (ii) we have $i_g(A) \subseteq i_g\ (A \cup B)$ and $i_g(B) \subseteq i_g\ (A \cup B)$. This implies $i_g(A) \cup i_g(B) \subseteq i_g(A \cup B)$.
4. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, from (ii) we have $i_g(A \cap B) \subseteq i_g(A)$ and $i_g(A \cap B) \subseteq i_g(B)$. This implies $i_g(A \cap B) \subseteq i_g(A) \cap i_g(B)$.
5. Since $i_g(A)$ is a g-open set in *X*, it follows that $i_g\ \left(i_g(A)\right) = i_g(A)$.

In the above Proposition 3.4 (iii) we note that $i_g(A) \cup i_g(B) \neq i_g(A \cup B)$.

Example 3.4

Let $X = \{x_1, x_2, x_3, x_4\}$ be generalized topological space with respect to generalized topology $\tau_g = \{\phi, X, \{x_1\}, \{x_3\}, \{x_1, x_3\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}\}$. Let us consider sets $A = \{x_1, x_2\}, B = \{x_3, x_4\}$. Then $i_g(A) = \{x_1\}$ and $i_g(B) = \{x_3\}$. Now $i_g(A) \cup i_g(B) = \{x_1, x_3\}$ and $i_g(A \cup B) = X$. Therefore $i_g(A) \cup i_g(B) \neq i_g(A \cup B)$.

In the above Proposition 3.4 (iv) we note that $i_g(A \cap B) \neq i_g(A) \cap i_g(B)$.

Example 3.5

Let $X = \{x_1, x_2, x_3, x_4\}$ be generalized topological space with respect to generalized topology $\tau_g = \{\phi, X, \{x_1, x_2\}, \{x_2, x_3, x_4\}\}$. Let us consider sets $A = \{x_1, x_2, x_3\}$, $B = \{x_2, x_3, x_4\}$. Then $i_g(A) = \{x_1, x_2\}$ and $i_g(B) = \{x_2, x_3, x_4\}$. Now $i_g(A) \cap i_g(B) = \{x_2\}$ and $i_g(A \cap B) = \phi$. Thus $i_g(A \cap B) \neq i_g(A) \cap i_g(B)$.

Proposition 3.7

Let (X, τ_g) be a generalized topological space and $\{A_\propto\}_{\propto \in \Lambda}$ be a family of subsets of X. Then

1. $\bigcup_{\propto \in \Lambda} i_g\ (A_\propto) \subseteq i_g\ (\bigcup_{\propto \in \Lambda} A_\propto)$.
2. $i_g\ (\bigcap_{\propto \in \Lambda} A_\propto) \subseteq \bigcap_{\propto \in \Lambda} i_g\ (A_\propto)$.

Proof

Similar to proof of Proposition 3.6 (iii) and (iv).

Proposition 3.8

Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Then

1. $i_g\ (X - A) = X - c_g(A)$.
2. $c_g\ (X - A) = X - i_g\ (A)$.

Proof

1. We have $X - c_g(A) = X - \bigcap_{\propto \in \Lambda}\{F_\propto : F_\propto \text{ is a g} - \text{colsed set in } X \text{ and } A \subseteq F_\propto\}$
   $= \bigcup_{\propto \in \Lambda} \{X - F_\propto : (X - F_\propto) \text{ is a g} - \text{open set in } X \text{ and } (X - F_\propto) \subseteq (X - A)\}$.
   $= i_g\ (X - A)$.
2. From (i), we have $X - c_g\ (X - A) = i_g(X - X - A = ig\ A$. Hence $X - ig\ A = cg\ X - A$.

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